Generic structures

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Part 1

Categories

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- a partial associative composition operation $\circ$ defined on arrows, where $f \circ g$ is defined $\iff$ the domain of g coincides with the domain of f.

Furthermore, for each $A \in \mathrm{Obj}(\mathfrak{K})$ there is an *identity* $\mathrm{id}_A \in \mathfrak{K}(A,A)$ satisfying $\mathrm{id}_A \circ g = g$ and $f \circ \mathrm{id}_A = f$ for $f \in \mathfrak{K}(A,X)$, $g \in \mathfrak{K}(Y,A)$, $X, Y \in \mathrm{Obj}(\mathfrak{K})$.

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Let $\vec{x}$ be a sequence in $\mathfrak{K}$. The colimit of $\vec{x}$ is a pair $\langle X, \{x_n^\infty\}_{n\in\mathbb{N}}\rangle$ with $x_n^\infty \colon X_n \to X$ satisfying:

1. $x_n^\infty = x_m^\infty \circ x_n^m$ for every $n < m$.
2. If $\langle Y, \{y_n^\infty\}_{n\in\mathbb{N}}\rangle$ with $y_n^\infty \colon X_n \to Y$ satisfies $y_n^\infty = y_m^\infty \circ y_n^m$ for every $n < m$ then there is a unique arrow $f \colon X \to Y$ satisfying $f \circ x_n^\infty = y_n^\infty$ for every $n \in \mathbb{N}$.

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More generally, after Odd's move finishing with an object A_{2k-1}, Eve chooses $A_{2k} \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_{2k-1}^{2k} \colon A_{2k-1} \to A_{2k}$.

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Next, Odd chooses $A_{2k+1} \in \mathrm{Obj}(\mathfrak{K})$ together with a $\mathfrak{K}$-arrow $a_{2k}^{2k+1} \colon A_{2k} \to A_{2k+1}$. And so on...

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The result of a play is a sequence $\vec{a}$:

$$A_0 \xrightarrow{a_0^1} A_1 \longrightarrow \cdots \longrightarrow A_{2k-1} \xrightarrow{a_{2k-1}^{2k}} A_{2k} \longrightarrow \cdots$$

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We say that $U \in \mathrm{Obj}(\mathfrak{L})$ is $\mathfrak{K}$-generic if Odd has a strategy in the Banach-Mazur game $\mathrm{BM}\,(\mathfrak{K})$ such that the colimit of the resulting sequence $\vec{a}$ is always isomorphic to U, no matter how Eve plays.

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A $\mathfrak{K}$-generic object, if exists, is unique up to isomorphism.

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Proof.

The rules for Eve and Odd are the same. $\square$

Examples

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Example

Let $\mathfrak{K}$ be the category of all finite acyclic graphs with embeddings.
Then the countable everywhere infinitely branching tree is $\mathfrak{K}$-generic.

Theorem (Urysohn, 1927)

There exists a unique Polish metric space $\mathbb{U}$ with the following property:

(E) *For every finite metric spaces $A \subseteq B$, every isometric embedding $e\colon A \to \mathbb{U}$ can be extended to an isometric embedding $f\colon B \to \mathbb{U}$.*

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Furthermore:

- *Every separable metric space embeds into $\mathbb{U}$.*
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Theorem

Let $\mathfrak{M}_{\mathrm{fin}}$ be the category of finite metric spaces with isometric embeddings.
Then the Urysohn space $\mathbb{U}$ is $\mathfrak{M}_{\mathrm{fin}}$-generic.

The amalgamation property

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Definition

We say that $\mathfrak{K}$ has amalgamations at $Z \in \mathrm{Obj}(\mathfrak{K})$ if for every $\mathfrak{K}$-arrows $f \colon Z \to X$, $g \colon Z \to Y$ there exist $\mathfrak{K}$-arrows $f' \colon X \to W$, $g' \colon Y \to W$ such that $f' \circ f = g' \circ g$.

$$\begin{tikzcd}
Y \arrow[r, "g"] & W \\
Z \arrow[u, "g"] \arrow[r, "f"'] & X \arrow[u, "f'"']
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$$\begin{tikzcd} Y \arrow[r, "g"] & W \\ Z \arrow[u, "g"] \arrow[r, "f"'] & X \arrow[u, "f'"'] \end{tikzcd}$$

We say that $\mathfrak{K}$ has the amalgamation property (AP) if it has amalgamations at every $Z \in \mathrm{Obj}(\mathfrak{K})$.

Theorem (Universality)

Assume $\mathfrak{K}$ has the AP and U is $\mathfrak{K}$-generic.
Then for every $X = \lim \vec{x}$, where $\vec{x}$ is a sequence in $\mathfrak{K}$, there exists an arrow

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Example

Let $\mathfrak{K}$ be the category of all finite linear graphs with embeddings. Then $\langle \mathbb{Z}, E \rangle$ is $\mathfrak{K}$-generic, where $xEy \iff |x - y| = 1$.
On the other hand, $\langle \mathbb{Z}, E \rangle \oplus \langle \mathbb{Z}, E \rangle \not\to \langle \mathbb{Z}, E \rangle$.

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2. For every $n \in \omega$, for every $\mathfrak{K}$-arrow $f\colon U_n \to Y$ there are $m > n$ and a $\mathfrak{K}$-arrow $g\colon Y \to U_m$ such that $g \circ f = u_n^m$.

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$$\begin{tikzcd} U_0 \arrow[r] & \cdots \arrow[r] & U_n \arrow[rr, "u_n^m"] \arrow[dr, "f"'] & & U_m \arrow[r] & \cdots \\ & & & Y \arrow[ur, "g"'] & & \end{tikzcd}$$

Theorem 1

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Proof.

$$\begin{tikzcd}
\cdots \arrow[r] & U_{n_0} \arrow[rr] \arrow[dr] & & U_{n_1} \arrow[rr] \arrow[dr] & & U_{n_2} \arrow[r] \arrow[dr] & \cdots \\
A_0 \arrow[ur] & & A_2 \arrow[ur] & & A_4 \arrow[ur] & & A_6
\end{tikzcd}$$

$\square$

Fraïssé categories

Definition

A Fraïssé category is a countable category $\mathfrak{K}$ satisfying:

1. For every $X, Y \in \mathrm{Obj}(\mathfrak{K})$ there is $U \in \mathrm{Obj}(\mathfrak{K})$ such that

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2. $\mathfrak{K}$ has the amalgamation property.

Theorem 2

Assume $\mathfrak{K} \subseteq \mathfrak{L}$ is such that every sequence in $\mathfrak{K}$ converges in $\mathfrak{L}$ and $\mathfrak{K}$ is a Fraïssé category. Then there exists a $\mathfrak{K}$-generic object in $\mathfrak{L}$.

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Proof.

Let $\mathbb{P}$ be the poset of all finite sequences in $\mathfrak{K}$, i.e., covariant functors from some $n \in \omega$ into $\mathfrak{K}$. The ordering is end-extension.

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Let

$$\mathscr{D} = \{D_{n,f} \colon n \in \omega,\ f \in \mathfrak{K}\} \cup \{E_{n,A} \colon n \in \omega,\ X \in \mathrm{Obj}(\mathfrak{K})\},$$

where

$$D_{n,f} = \{\vec{x} \in \mathbb{P} \colon X_n = \mathrm{dom}(f) \implies (\exists m > n)(\exists g)\ g \circ f = x_n^m\},$$

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Let $\vec{u}$ be the sequence coming from a $\mathscr{D}$-generic filter/ideal. Then $\vec{u}$ is Fraïssé, therefore $U = \lim \vec{u}$ is $\mathfrak{K}$-generic. $\square$

Fraïssé theory

Definition

A Fraïssé class is a class of finite models of a fixed countable language satisfying:

(H) For every $A \in \mathscr{F}$, every model isomorphic to a submodel of A is in $\mathscr{F}$.

(JEP) Every two models from $\mathscr{F}$ embed into a single model from $\mathscr{F}$.

(AP) $\mathscr{F}$ has the amalgamation property for embeddings.

(CMT) $\mathscr{F}$ has countably many isomorphic types.

Theorem (Fraïssé, 1954)

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Conversely, if U is a countable homogeneous model then the class of all models isomorphic to finite submodels of U is Fraïssé.

The model U is called the Fraïssé limit of $\mathscr{F}$.

More examples

The Cantor set

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$$\begin{tikzcd}
 & T \arrow[dd, two heads, "p"] \\
K \arrow[ur, "g"] \arrow[dr, "f"'] & \\
 & S
\end{tikzcd}$$

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$$\begin{tikzcd} & Y \arrow[dd, two heads, "p"] \\ K \arrow[ur, "g"] \arrow[dr, "f"'] & \\ & X \end{tikzcd}$$

Theorem (Bielas, Walczyńska, K.)

Let 2^ω denote the Cantor set. A continuous mapping $\eta \colon K \to 2^\omega$ is $\mathfrak{K}_K$-generic $\iff$ η is a topological embedding and $\eta[K]$ is nowhere dense in 2^ω.

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Corollary (Knaster & Reichbach 1953)

Let $h \colon A \to B$ be a homeomorphism between closed nowhere dense subsets of 2^ω. Then there exists a homeomorphism $H \colon 2^\omega \to 2^\omega$ such that

$$H \restriction A = h.$$

The Gurarii space

Theorem (Gurarii 1966)

There exists a separable Banach space $\mathbb{G}$ with the following property.

(G) *For every $\varepsilon > 0$, for every finite-dimensional normed spaces $E \subseteq F$, for every linear isometric embedding $e \colon E \to \mathbb{G}$ there exists a linear ε-isometric embedding $f \colon F \to \mathbb{G}$ such that $f \restriction E = e$.*

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Elementary proof: Solecki & K. 2013.

Theorem

The Gurarii space $\mathbb{G}$ is generic over the category $\mathfrak{B}_{\mathsf{fd}}$ of finite-dimensional normed spaces with linear isometric embeddings.

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Key Lemma (Solecki & K.)

Let X, Y be finite-dimensional normed spaces, let $f\colon X \to Y$ be an ε-isometry with $0 < \varepsilon < 1$. Then there exist a finite-dimensional normed space Z and isometric embeddings $i\colon X \to Z$, $j\colon Y \to Z$ such that

$$\|i - j \circ f\| \leqslant \varepsilon.$$

The pseudo-arc

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The pseudo-arc is $\mathfrak{I}$-generic.

Part 2

Weak Fraïssé sequences

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A sequence $\vec{u} \colon \omega \to \mathfrak{K}$ is a weak Fraïssé sequence if it satisfies the following conditions:

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2. For every $n \in \omega$ there exists $n^* > n$ such that for every $\mathfrak{K}$-arrow $f\colon U_{n^*} \to Y$ there are $m > n^*$ and a $\mathfrak{K}$-arrow $g\colon Y \to U_m$ with $g \circ f \circ u_n^{n^*} = u_n^m$.

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A sequence $\vec{u}\colon \omega \to \mathfrak{K}$ is a weak Fraïssé sequence if it satisfies the following conditions:

1. For every $A \in \mathrm{Obj}(\mathfrak{K})$ there is n such that $\mathfrak{K}(A, U_n) \neq \emptyset$.
2. For every $n \in \omega$ there exists $n^* > n$ such that for every $\mathfrak{K}$-arrow $f\colon U_{n^*} \to Y$ there are $m > n^*$ and a $\mathfrak{K}$-arrow $g\colon Y \to U_m$ with $g \circ f \circ u_n^{n^*} = u_n^m$.

$$\begin{tikzcd}
\cdots \arrow[r] & U_n \arrow[r] & U_{n^*} \arrow[rr] \arrow[dr, "f"'] & & U_m \arrow[r] & \cdots \\
& & & Y \arrow[ur, "g"'] & &
\end{tikzcd}$$

Theorem

Assume $\vec{u}$ is a weak Fraïssé sequence in $\mathfrak{K}$ and $U = \lim \vec{u}$. Then U is $\mathfrak{K}$-generic.

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Proof.

$$
\begin{tikzcd}
\cdots \arrow[r] & U_{n_0} \arrow[r] & U_{n_0^*} \arrow[r] \arrow[d] & U_{n_1} \arrow[r] & U_{n_1^*} \arrow[r] \arrow[d] & U_{n_2} \arrow[r] & \cdots \\
A_0 \arrow[ur] & & A_2 \arrow[ur] & & A_4 \arrow[ur] & &
\end{tikzcd}
$$

$\square$

Weakenings of amalgamation

Definition

We say that $\mathfrak{K}$ has the cofinal amalgamation property (CAP) if for every $Z \in \mathrm{Obj}(\mathfrak{K})$ there is a $\mathfrak{K}$-arrow $e\colon Z \to Z'$ such that $\mathfrak{K}$ has amalgamations at Z'.

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Definition (Ivanov 1999; Kechris & Rosendal 2007; Kruckman 2016)

We say that $\mathfrak{K}$ has the weak amalgamation property (WAP) if for every $Z \in \mathrm{Obj}(\mathfrak{K})$ there is a $\mathfrak{K}$-arrow $e \colon Z \to Z'$ such that for every $\mathfrak{K}$-arrows $f \colon Z' \to X$, $g \colon Z' \to Y$ there exist $\mathfrak{K}$-arrows $f' \colon X \to W$, $g' \colon Y \to W$ such that $f' \circ f \circ e = g' \circ g \circ e$.

CAP and WAP

$$\begin{tikzcd}
 & Y \arrow[r, "g'"] & W \\
 & Z' \arrow[u, "g"] \arrow[r, "f"'] & X \arrow[u, "f'"'] \\
Z \arrow[ur, "e"] & &
\end{tikzcd}$$

CAP and WAP

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\begin{tikzcd}
& Y \arrow[r, "g'"] & W \\
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Z \arrow[ur, "e"] & &
\end{tikzcd}
$$

Proposition

Finite graphs of vertex degree $\leqslant 2$ have the CAP.

Weak injectivity

Definition

An object $V \in \mathrm{Obj}(\mathfrak{L})$ is weakly $\mathfrak{K}$-injective if

- every $\mathfrak{K}$-object has an $\mathfrak{L}$-arrow into V, and
- for every $\mathfrak{L}$-arrow $e \colon A \to V$ there exists a $\mathfrak{K}$-arrow $i \colon A \to B$ such that for every $\mathfrak{K}$-arrow $f \colon B \to Y$ there is an $\mathfrak{L}$-arrow $g \colon Y \to V$ satisfying $g \circ f \circ i = e$.

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$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "e"'] & B \arrow[r, "f"] & Y \arrow[dll, dashed, "g"] \\ V & & \end{tikzcd}$$

Theorem (Krawczyk & K. 2016)

Let $\mathfrak{K}$ be a countable directed category of finitely generated models with embeddings.

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(b) *$\mathfrak{K}$ has the WAP.*

Theorem (Krawczyk & K. 2016)

Let $\mathfrak{K}$ be as above and let U be a countably generated model. The following properties are equivalent:

(a) *U is $\mathfrak{K}$-generic.*

(b) *Eve does not have a winning strategy in $\mathrm{BM}(\mathfrak{K}, U)$.*

(c) *U is weakly $\mathfrak{K}$-injective.*

The first example of a weak Fraïssé class with no CAP

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- J.-F. PABION, *Relations préhomogènes*, C. R. Acad. Sci. Paris Sér. A-B 274 (1972) A529–A531.

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A quote from Pabion's paper:

3º M. Pouzèt m'a communiqué l'exemple suivant de relation uniformément préhomogène et non pseudo-homogène. Sur Q, définir R (x, y, z) par $x < y$, $x < z$ et $y \neq z$.

(*) Séance du 7 février 1972.

(1) J. P. Calais, *Comptes rendus*, 265, série A, 1967, p. 2.

(2) R. Fraïssé, *Cours de Logiques mathématiques*, I, Gauthiers-Villars, Paris, 1967, deuxième édition 1971.

(3) G. Kreisel, *The theory of models*, North-Holland, 1970.

(4) P. Linsdtrom, *Theoria*, 30, 1964, p. 183-196.

(5) R. L. Vaught, *Bull. Amer. Math. Soc.*, 69, p. 229-313.

Université Claude Bernard,
Mathématiques,
43, *boulevard du Onze-Novembre* 1918,
69-*Villeurbanne, Rhône.*

Weak Fraïssé theory

Definition

A weak Fraïssé class is a class $\mathscr{F}$ of finitely generated models of a fixed countable signature, closed under isomorphisms, having with many types, satisfying (JEP) and (WAP).

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Let $\mathscr{F}$ be a weak Fraïssé class. Then there exists a unique countable model U that is weakly $\mathscr{F}$-injective and weakly homogeneous. Furthermore, U is $\mathscr{F}$-generic.

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Theorem

Let $\mathscr{F}$ be a weak Fraïssé class. Then there exists a unique countable model U that is weakly $\mathscr{F}$-injective and weakly homogeneous. Furthermore, U is $\mathscr{F}$-generic.
Conversely, given a countable weakly homogeneous model M, its age

$$\mathscr{F} = \{A \colon A \text{ is finitely generated and embeddable into } M\}$$

is a weak Fraïssé class.

Definition

A structure M is weakly homogeneous if for every finitely generated substructure $A \subseteq M$ there is a bigger finitely generated substructure $B \subseteq M$ containing A such that every embedding $e \colon A \to M$ extendable to B extends to an automorphism of M.

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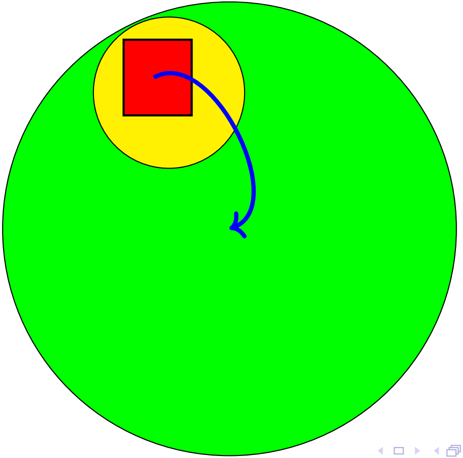

Some references

1. W. KUBIŚ, *Weak Fraïssé categories*, preprint, arXiv:1712.03300
2. A. KRAWCZYK, W. KUBIŚ, *Games on finitely generated structures*, preprint, arXiv:1701.05756

Some references

1. W. KUBIŚ, *Weak Fraïssé categories*, preprint, arXiv:1712.03300
2. A. KRAWCZYK, W. KUBIŚ, *Games on finitely generated structures*, preprint, arXiv:1701.05756

- A. IVANOV, *Generic expansions of ω-categorical structures and semantics of generalized quantifiers*, J. Symbolic Logic 64 (1999) 775–789
- A.S. KECHRIS, C. ROSENDAL, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proc. Lond. Math. Soc. (3) 94 (2007) 302–350
- Z. KABLUCHKO, K. TENT, *On weak Fraisse limits*, preprint, arXiv:1711.09295

Theorem (Krawczyk, Kruckman, Panagiotopoulos, K. 2018)

There exist continuum many hereditary weak Fraïssé classes of finite graphs without the cofinal AP.

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Example

Let $\mathscr{G}$ be the class of all finite acyclic graphs in which no two vertices of degree > 2 are adjacent.
Then $\mathscr{G}$ is a weak Fraïssé class failing the CAP.

Peano continua

Theorem (Bartoš, K. 2018)

Let $\mathfrak{K}$ be a class of non-degenerate Peano continua treated as a category with all continuous surjections. Then the pseudo-arc is $\mathfrak{K}^{\mathrm{op}}$-generic.

($\mathfrak{K}^{\mathrm{op}}$ is the category opposite to $\mathfrak{K}$.)

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($\mathfrak{K}^{\mathrm{op}}$ is the category opposite to $\mathfrak{K}$.)

Theorem (Kwiatkowska, K. 2017)

The Poulsen simplex is generic over the (opposite) category of finite-dimensional simplices with affine surjections.

Automorphism groups of Fraïssé limits

Question (Eric Jaligot, 2007)

Let M be a countable homogeneous structure. Is it always true that the group $\mathrm{Aut}(M)$ contains isomorphic copies of all groups of the form $\mathrm{Aut}(X)$, where X is a substructure of M?

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Let M be a countable homogeneous structure. Is it always true that the group $\mathrm{Aut}(M)$ contains isomorphic copies of all groups of the form $\mathrm{Aut}(X)$, where X is a substructure of M?

If this is the case, we shall say that $\mathrm{Aut}(M)$ is universal.

Uniform homogeneity

Definition (Kuzeljević, K. 2018)

A structure M is uniformly homogeneous if

1. M is homogeneous and
2. for every finite substructure $A \subseteq M$ there exists an extension operator $e_A \colon \operatorname{Aut}(A) \to \operatorname{Aut}(M)$ such that

$$e_A(g \circ h) = e_A(g) \circ e_A(h)$$

for every $g, h \in \operatorname{Aut}(A)$.

A homogeneous digraph that is not uniformly homogeneous

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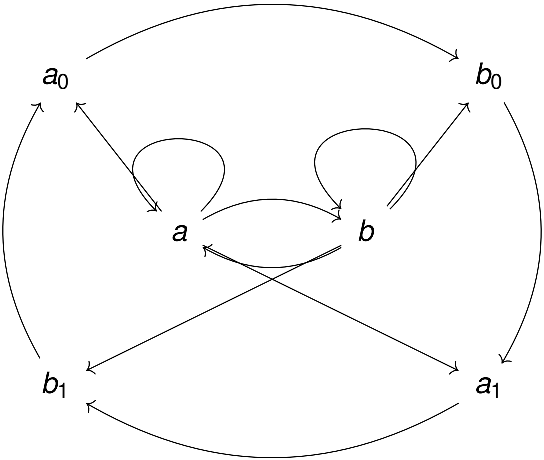

Katětov functors

Definition

Let $\mathscr{F}$ be a class of finite structures of the same type and let M be a countable homogeneous structure such that every $A \in \mathscr{F}$ embeds into M and every finite substructure of M is isomorphic to some $A \in \mathscr{F}$. A Katětov functor is a pair $\langle K, \eta \rangle$ such that K assigns to each embedding $e \colon A \to B$ with $A, B \in \mathscr{F}$ an embedding $K(e) \colon M \to M$, η assigns to each $A \in \mathscr{F}$ an embedding $\eta_A \colon A \to M$. Furthermore, K is a functor, i.e., $K(\mathrm{id}_A) = \mathrm{id}_M$, $K(e \circ f) = K(e) \circ K(f)$, and the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "e"'] & M \arrow[d, "K(e)"] \\ B \arrow[r, "\eta_B"'] & M \end{tikzcd}$$

for every embedding $e \colon A \to B$ with $A, B \in \mathscr{F}$.

Theorem (Mašulović & K.)

Assume $\langle \mathscr{F}, M \rangle$ admits a Katětov functor. Then for every substructure X of M there exists a topological group embedding

$$e_X \colon \operatorname{Aut}(X) \to \operatorname{Aut}(M).$$

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Claim (☹)

Most of the well known homogeneous relational structures admit a Katětov functor.

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Theorem (Grebík 2018)

Let $\mathscr{F}$ be the class of all finite complete graphs whose edges are colored, the set of colors is countably infinite, and there is no monochromatic triangle.
Then $\mathscr{F}$ is a Fraïssé class with no Katětov functor, yet the automorphism group of its limit is universal.

Part 3

Non-universal automorphism groups

Theorem (Shelah & K. 2018)

There exists a countable homogeneous relational structure E such that:

- *every finite group embeds into* $\mathrm{Aut}(E)$,
- S_∞ *does not embed into* $\mathrm{Aut}(E)$,
- $S_\infty \approx \mathrm{Aut}(X)$ *for some* $X \subseteq E$.

Furthermore, E is not uniformly homogeneous.

Sketch of proof.

Let $\mathfrak{M}$ be the class of all finite consumer-goods models. Namely, $M \in \mathfrak{M}$ consists of a set C^M of consumers together with a set G^M of goods, and each consumer $c \in C^M$ has a strict preference relation $<_c$ which is just a linear ordering of G^M.

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Claim

$\mathfrak{M}$ is a Fraïssé class.

Let U be the Fraïssé limit of $\mathfrak{M}$. Define

$$G_c = \{h \in \mathrm{Aut}(U) \colon h(c) = c\},$$

where $c \in C^U$ is a fixed consumer. Then G_c embeds into $\mathrm{Aut}(\mathbb{Q}, <)$, therefore it is torsion-free.

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Clearly, G_c has a countable index in $\mathrm{Aut}(U)$.

Sketch of proof (continued).

Claim

Assume G is a group, H is a subgroup of G of index $< \mathfrak{c}$ in G, such that $x^2 \neq 1$ for every $x \in H \setminus \{1\}$. Then $S_\infty \not\to G$.

Sketch of proof (continued).

Claim

Assume G is a group, H is a subgroup of G of index $< \mathfrak{c}$ in G, such that $x^2 \neq 1$ for every $x \in H \setminus \{1\}$. Then $S_\infty \not\to G$.

Proof.

The group $\mathbb{Z}_2^\omega$ embeds into S_∞. □

Sketch of proof (continued).

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Assume G is a group, H is a subgroup of G of index $< \mathfrak{c}$ in G, such that $x^2 \neq 1$ for every $x \in H \setminus \{1\}$. Then $S_\infty \not\hookrightarrow G$.

Proof.

The group $\mathbb{Z}_2^\omega$ embeds into S_∞. □

Finally, $S_\infty \approx \mathrm{Aut}(M)$, where $M = C^U$ (so $C^M = C^U$ and $G^M = \emptyset$). By the Claim above, $S_\infty \not\hookrightarrow \mathrm{Aut}(U)$. □

Theorem (Shelah & K. 2018)

There exists a countable homogeneous relational structure M such that:

- $\mathrm{Aut}(M)$ *is torsion-free,*
- *for every $n \in \mathbb{N}$ there is a finite $A \subseteq M$ with $S_n \approx \mathrm{Aut}(A)$.*

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Let $\mathscr{F}$ be the class of all finite bipartite graphs with the left-hand side linearly ordered.

Claim

$\mathscr{F}$ is a Fraïssé class.

Let M be the Fraïssé limit of $\mathscr{F}$. Then M is as required. □

Question (Tsankov)

Does there exist a transitive Fraïssé limit whose automorphism group is universal?

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Some more references

- R. Fraïssé, *Sur lextension aux relations de quelques propriétés des ordres*, Ann. Sci. Ecole Norm. Sup. (3) 71 (1954) 363–388
- J. Grebík, *An example of a Fraïssé class without a Katětov functor*, Appl. Categor. Struct. 26 (2018) 1–6
- E. Jaligot, *On stabilizers of some moieties of the random tournament*, Combinatorica 27 (2007) 129–133
- W. Kubiś, D. Mašulović, *Katětov functors*, Applied Categorical Structures 25 (2017) 569–602
- W. Kubiś, S. Shelah, *Homogeneous structures with non-universal automorphism groups*, preprint, arXiv:1811.09650

Uncountable Fraïssé limits

Weak Fraïssé sequences again

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Definition

A weak Fraïssé sequence sequence is a sequence $\vec{u}\colon \kappa \to \mathfrak{K}$, where κ is an infinite cardinal, satisfying the following conditions:

1. For every $A \in \mathrm{Obj}(\mathfrak{K})$ there is $\alpha < \kappa$ such that $\mathfrak{K}(A, U_\alpha) \neq \emptyset$.

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2. For every α there exists $\alpha^* > \alpha$ such that for every $\mathfrak{K}$-arrow $f\colon U_{n^*} \to Y$ there are $\beta > \alpha^*$ and a $\mathfrak{K}$-arrow $g\colon Y \to U_\beta$ with $g \circ f \circ u_\alpha^{\alpha^*} = u_\alpha^\beta$.

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$$\begin{tikzcd}
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\end{tikzcd}$$

Theorem

Assume $\kappa \geqslant \aleph_0$ is a regular cardinal, $\vec{u}$ is a continuous weak Fraïssé sequence in $\mathfrak{K}$ with $U = \lim \vec{u}$ in $\mathfrak{L}$. Then U is $\mathfrak{K}$-generic.

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Proof.

$$\begin{tikzcd}
\cdots \arrow[r] & U_{\alpha_0} \arrow[r] & U_{\alpha_0^*} \arrow[r] \arrow[d] & U_{\alpha_1} \arrow[r] & U_{\alpha_1^*} \arrow[r] \arrow[d] & U_{\alpha_2} \arrow[r] & \cdots \\
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Theorem

Assume κ is a regular cardinal, sequences of length $< \kappa$ have colimits in $\mathfrak{K}$, and $\mathfrak{K}$ is dominated by a subcategory of size $\leqslant \kappa$. Then there exists a continuous weak Fraïssé sequence in $\mathfrak{K}$.

Theorem

Let κ be a regular cardinal, let $\vec{u}$ be a weak Fraïssé sequence in $\mathfrak{K}$ with

$$U = \lim \vec{u} \in \mathrm{Obj}(\mathfrak{L}).$$

Assume every $\mathfrak{L}$-object is the colimit of a κ-sequence in $\mathfrak{K}$.

Theorem

Let κ be a regular cardinal, let $\vec{u}$ be a weak Fraïssé sequence in $\mathfrak{K}$ with

$$U = \lim \vec{u} \in \mathrm{Obj}(\mathfrak{L}).$$

Assume every $\mathfrak{L}$-object is the colimit of a κ-sequence in $\mathfrak{K}$.
If $\mathfrak{K}$ has the amalgamation property then

- *$\vec{u}$ is Fraïssé,*
- *U is $\mathfrak{K}$-homogeneous,*
- *U is $\mathfrak{L}$-cofinal, that is, $\mathfrak{L}(X, U) \neq \emptyset$ for every $X \in \mathrm{Obj}(\mathfrak{L})$.*

Example

Let $\mathfrak{K}$ be the category of all countable linearly ordered sets with left-invertible increasing mappings.

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Let

$$\mathrm{Obj}(\mathscr{F}) = \{\mathbb{Q}\}, \qquad \mathscr{F}(\mathbb{Q},\mathbb{Q}) = \{\mathrm{id}_{\mathbb{Q}}, e\},$$

where $e\colon \mathbb{Q} \to \mathbb{Q}$ is isomorphic to the canonical embedding of $\mathbb{Q}$ into $\mathbb{Q} \cdot \mathbb{Q}$, defined by $r \mapsto \langle r, 0\rangle$.

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Let

$$\mathrm{Obj}(\mathscr{F}) = \{\mathbb{Q}\}, \qquad \mathscr{F}(\mathbb{Q},\mathbb{Q}) = \{\mathrm{id}_{\mathbb{Q}}, e\},$$

where $e\colon \mathbb{Q} \to \mathbb{Q}$ is isomorphic to the canonical embedding of $\mathbb{Q}$ into $\mathbb{Q} \cdot \mathbb{Q}$, defined by $r \mapsto \langle r, 0\rangle$.

Claim

$\mathfrak{K}$ has the AP and $\mathscr{F}$ is dominating in $\mathfrak{K}$.

References:

1. W. Kubiś, *Fraïssé sequences: category-theoretic approach to universal homogeneous structures*, Ann. Pure Appl. Logic 165 (2014) 1755–1811

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3. T. Irwin, S. Solecki, *Projective Fraïssé limits and the pseudo-arc*, Trans. Amer. Math. Soc. 358 (2006) 3077–3096

Continuous Fraïssé limits

The setup: Normed categories

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A normed category is a category $\mathfrak{L}$ enriched over metric spaces, endowed with a function $\mu\colon \mathfrak{L} \to [0, +\infty]$, called a norm, satisfying

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Furthermore, μ is uniformly continuous, namely, for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$|\mu(f) - \mu(g)| < \varepsilon$$

whenever $\varrho(f, g) < \delta$.

Lemma

$\mu(h) = \mu(h^{-1})$ *whenever h is an isomorphism.*

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Proof.

Let $x = \mathrm{dom}(h)$. By ($\mathscr{N}3$) and ($\mathscr{N}1$), $\mu(h) \leqslant \mu(\mathrm{id}_x) + \mu(h^{-1}) = \mu(h^{-1})$.
Replacing h by h^{-1} leads to $\mu(h^{-1}) \leqslant \mu(h)$. $\square$

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Definition

Isomorphisms of norm 0 will be called isometries.

A motivating example

Example

Let $\mathfrak{L}$ be the category of Banach spaces with non-expansive linear operators. Define

$$\mu(f) = \inf\{\|i - j \circ f\| : i, j \text{ are isometric embeddings compatible with } f\}.$$

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Lemma (Solecki & K.)

If f is an ε-isometry, $0 < \varepsilon < 1$ then $\mu(f) \leqslant \varepsilon$.

It is easy to see that μ is a norm on $\mathfrak{L}$.

The setup (continued)

We assume $\mathfrak{K}$ is a full subcategory of a normed category $\mathfrak{L}$.

$$\mathfrak{L}_0 := \{f \in \mathfrak{L} \colon \mu(f) = 0\}, \quad \mathfrak{K}_0 := \mathfrak{K} \cap \mathfrak{L}_0.$$

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Definition

We say that $\mathfrak{K}$ has the almost amalgamation property if for every $\mathfrak{K}_0$-arrows $f \colon Z \to X$, $g \colon Z \to Y$ for every $\varepsilon > 0$ there are $\mathfrak{K}_0$-arrows $f' \colon X \to W$, $g' \colon Y \to W$ satisfying

$$\varrho(f' \circ f, g' \circ g) < \varepsilon.$$

Definition

We say that the norm μ is stable if for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $f \in \mathfrak{K}$ with $\mu(f) < \delta$ there are compatible $i, j \in \mathfrak{K}_0$ such that

$$\varrho(i, j \circ f) < \varepsilon.$$

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "i"] \arrow[rr, "f"'] & & Y \arrow[ul, "j"'] \end{tikzcd}$$

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2. *U is approximately $\mathfrak{K}_0$-homogeneous. Namely, given $\varepsilon > 0$, given $\mathfrak{L}_0$-arrows $i\colon A \to U$, $j\colon A \to U$ there is an isometry $h\colon U \to U$ such that $\varrho(h \circ i, j) < \varepsilon$.*

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Source:

- W. Kubiś, *Metric-enriched categories and approximate Fraïssé limits*, preprint, [arXiv:1210.6506]

Selected Applications

- W. Kubiś, A. Kwiatkowska, *The Lelek fan and the Poulsen simplex as Fraïssé limits*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM 111 (2017) 967–981
- J. Garbulińska-Wegrzyn, W. Kubiś, *A universal operator on the Gurarii space*, Journal of Operator Theory 73 (2015) 143–158
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Thank you for your attention!